On Elliptic Lax Systems on the Lattice and a Compound Theorem for Hyperdeterminants

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Abstract

A general elliptic $N \times N$ matrix Lax scheme is presented, leading to two classes of elliptic lattice systems, one which we interpret as the higher-rank analogue of the Landau-Lifschitz equations, while the other class we characterize as the higher-rank analogue of the lattice Krichever-Novikov equation (or Adler's lattice). We present the general scheme, but focus mainly of the latter type of models. In the case $N = 2$ we obtain a novel Lax representation of Adler's elliptic lattice equation in its so-called 3-leg form. The case of rank $N = 3$ is analysed using Cayley's hyperdeterminant of format $2 \times 2 \times 2$, yielding a multi-component system of coupled 3-leg quad-equations.

1 Introduction

Adler's lattice equation, [1], is an integrable lattice version of the Krichever-Novikov (KN) equation, [15], i.e. of the nonlinear evolution equation

$$u_t = \frac{1}{4}\left(u_{xxx} + \frac{3}{2}\,\frac{r(u) - u_{xx}^2}{u_x}\right) \ , \tag{1.1}$$

in which $r(u) = 4u^3 - g_2 u - g_3$ is the polynomial associated with a Weierstrass elliptic curve (or more generally an arbitrary quartic polynomial). This lattice equation, which was obtained as the permutability condition for the Bäcklund transformations for (1.1), can be written in the form[1]:

$$\begin{aligned} &A\,[(u-b)(\widehat{u}-b) - (a-b)(c-b)]\left[(\widetilde{u}-b)(\widehat{\widetilde{u}}-b) - (a-b)(c-b)\right] \\ &+B\,[(u-a)(\widetilde{u}-a) - (b-a)(c-a)]\left[(\widehat{u}-a)(\widehat{\widetilde{u}}-a) - (b-a)(c-a)\right] = \\ &= ABC(a-b) \ , \end{aligned} \tag{1.2}$$

cf. [16], where $u = u(n,m)$ is the dependent variable, with the shifted variables $\widetilde{u} = u(n+1,m)$, $\widehat{u} = u(n,m+1)$ and $\widehat{\widetilde{u}} = u(n+1,m+1)$ defining the different values of u at the vertices around an elementary plaquette, cf. Figure 1. The $\mathfrak{a}$,$\mathfrak{b}$ in Figure 1 are lattice parameters associated with the grid size, and in this elliptic equation they are points $\mathfrak{a} = (a,A)$, $\mathfrak{b} = (b,B)$, together with $\mathfrak{c} = (c,C)$, on a Weierstrass elliptic curve, i.e.

$$A^2 = r(a) \equiv 4a^3 - g_2 a - g_3 \quad , \quad B^2 = r(b) \quad , \quad C^2 = r(c) \ , \tag{1.3}$$

which can parametrised in terms of the Weierstrass $\wp$−function as follows:

$$(a,A) = (\wp(\alpha), \wp'(\alpha)), \quad (b,B) = (\wp(\beta), \wp'(\beta)), \quad (c,C) = (\wp(\gamma), \wp'(\gamma)) \ , \tag{1.4}$$

where α and β are the corresponding uniformising parameters and where $\gamma = \beta - \alpha$. The parameters $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$ are related through the addition formulae on the elliptic curve:

$$\begin{aligned} A(c-b) &= C(a-b) - B(c-a), \\ a+b+c &= \frac{1}{4}\left(\frac{A+B}{a-b}\right)^2 \ . \end{aligned} \tag{1.5}$$

[1]Note that in the original paper [1] the equation was written in a slightly different form with rather complicated expressions for the coefficients given in terms of the moduli g_2 and g_3 of the Weierstrass curve.

Furthermore, we use the notation for the lattice shifts

$$u \xrightarrow{\alpha} \widetilde{u} \ , \qquad u \xrightarrow{\beta} \widehat{u}$$

being the elementary shifts on a quadrilateral lattice, each being associated with the lattice parameters (a, A) respectively (b, B), with the equation (1.2) expressing the condition for commutativity of these shifts as expressed through the diagram:

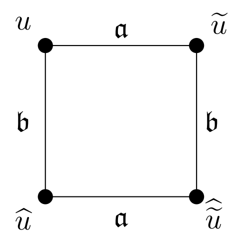

Figure 1: Configuration of lattice points in the lattice equation (1.2).

A Lax pair for Adler's equation was given in [16], and the equation reemerged in [3] as the top equation in the ABS list of affine-linear quadrilateral equations, where it was renamed Q4. The key integrability characteristic of Adler's equation is its *multidimensional consistency*, [18, 8], which in the case of Adler's equation can be made manifest through its so-called 3-leg form, cf. [3]:

$$\frac{\sigma(\widetilde{\xi}-\xi+\alpha)\,\sigma(\widetilde{\xi}+\xi-\alpha)}{\sigma(\widetilde{\xi}-\xi-\alpha)\,\sigma(\widetilde{\xi}+\xi+\alpha)}\,\frac{\sigma(\widehat{\xi}-\xi-\beta)\,\sigma(\widehat{\xi}+\xi+\beta)}{\sigma(\widehat{\xi}-\xi+\beta)\,\sigma(\widehat{\xi}+\xi-\beta)}=\frac{\sigma(\widehat{\widetilde{\xi}}-\xi-\gamma)\,\sigma(\widehat{\widetilde{\xi}}+\xi+\gamma)}{\sigma(\widehat{\widetilde{\xi}}-\xi+\gamma)\,\sigma(\widehat{\widetilde{\xi}}+\xi-\gamma)} \tag{1.6}$$

in which the uniformising variable $\xi = \xi(n, m)$ is now the dependent variable of the equation, related to the original variable u of the rational form (1.2) of the equation through the identification $u = \wp(\xi)$. The connection between rational and elliptic form of the equation parallels that of the KN equation, which in its (original) elliptic form reads:

$$\xi_t = \frac{1}{4}\left(\xi_{xxx} + \frac{3}{2}\,\frac{1-\xi_{xx}^2}{\xi_x} - 6\wp(2\xi)\,\xi_x^3\right) \ . \tag{1.7}$$

We note in passing that there are alternative forms for Adler's equation based on different choices of the underlying elliptic curve. Thus, if one could consider (1.2) to be the Weierstrass form of the equation (with parameters on a Weierstrass elliptic curve (1.3)), the equation in Jacobi form (due to Hietarinta, [13]) reads:

$$Q(v,\widetilde{v},\widehat{v},\widehat{\widetilde{v}}) = p(v\widetilde{v}+\widehat{v}\widehat{\widetilde{v}}) - q(v\widehat{v}+\widetilde{v}\widehat{\widetilde{v}}) - r(\widetilde{v}\widehat{v}+v\widehat{\widetilde{v}}) + pqr(1+v\widetilde{v}\widehat{v}\widehat{\widetilde{v}}) = 0 \tag{1.8}$$

where the dependent variable v is related to u of (1.2) through a fractional linear transformation, and where the parameters (p,P), (q,Q) and (r,R) are now points on a Jacobi type elliptic curve:

$$\Gamma: \quad X^2 \equiv x^4 - \gamma x^2 + 1, \qquad \gamma^2 = k + 1/k, \tag{1.9}$$

with modulus k. They can be parametrised in terms of Jacobi elliptic function as follows:

$$\begin{aligned}&\mathfrak{p} = (p,P) = (\sqrt{k}\,\mathrm{sn}(\alpha;\mathrm{k}), \mathrm{sn}'(\alpha;\mathrm{k})), \qquad \mathfrak{q} = (\mathrm{q},\mathrm{Q}) = (\sqrt{\mathrm{k}}\,\mathrm{sn}(\beta;\mathrm{k}), \mathrm{sn}'(\beta;\mathrm{k})),\\ &\mathfrak{r} = (r,R) = (\sqrt{k}\,\mathrm{sn}(\alpha-\beta;\mathrm{k}), \mathrm{sn}'(\alpha-\beta;\mathrm{k}))\ .\end{aligned} \tag{1.10}$$

Many interesting results were established for the latter form of the equation, notably explicit expressions for the (doubly elliptic) N-soliton solutions, [5], however for the sake of the present paper we will concentrate once again on the Weierstrass form of the equation.

In the present paper we propose a general elliptic Lax scheme of rank N, which is inspired by a novel Lax representation of Adler's lattice equation. This Lax scheme leads to two distinct classes of systems which we coin as being "of Landau-Lifschitz type" (or spin-nonzero case) and as "of Krichever-Novikov type" (or spin-zero case). We present general results for both classes in section 2, but then focus in the remainder of the paper on the Krichever-Novikov class of Lax systems. In that case for $N=2$ we show that the scheme amounts to a novel Lax representation for Adler's lattice equation, which yields the equation directly in 3-leg form (this in contrast with the lax pair constructed in [16] from multidimensional consistency). Notably in the rank $N=3$ case the analysis of the compatibility condition exploits a (to our knowledge novel) *compound theorem* for Caley's hyperdeterminants of format $2\times 2\times 2$, [9], a result which may have some significance in its own right. We conjecture that the resulting rank 3 lattice system may be regarded as a discrete analogue of a rank 3 Krichever-Novikov type of differential system that was constructed by Mokhov in [20].

2 General Elliptic Lax Scheme

Consider the Lax pair of the form:

$$\widetilde{\chi}_\kappa = L_\kappa \, \chi_\kappa \ , \tag{2.1a}$$

$$\widehat{\chi}_\kappa = M_\kappa \, \chi_\kappa \ . \tag{2.1b}$$

defining horizontal and vertical shifts of the vector function χ_κ, according to the diagram: where the vectors χ are located at the vertices of the quadrilateral and in which the

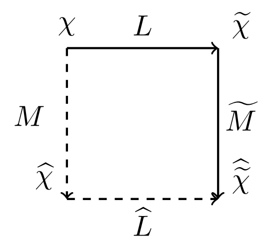

Figure 2: Lax compatibility condition (2.4).

matrices L and M are attached to the edges linking the vertices. The matrices L_κ andM_κ can be taken of the form;

$$(L_\kappa)_{i,j} = \Phi_{N\kappa}(\widetilde{\xi}_i - \xi_j - \alpha) h_j \ , \tag{2.2a}$$

$$(M_\kappa)_{i,j} = \Phi_{N\kappa}(\widehat{\xi}_i - \xi_j - \beta) k_j \ , \tag{2.2b}$$

$$(i, j = 1, \dots, N)$$

in which Φ_κ denotes the (truncated) Lamé function

$$\Phi_\kappa(\xi) \equiv \frac{\sigma(\xi + \kappa)}{\sigma(\xi)\sigma(\kappa)} \tag{2.3}$$

with σ denoting the Weierstrass σ-function and the variables $\xi_i = \xi_i(n, m)$, $(i = 1, \dots, N)$, are the main dependent variables. As before α and β denote the uniformised lattice parameters (as in (1.4)), while κ is the (uniformised) spectral parameter. In (2.2), the coefficients h_j, k_j, are some functions of the variables ξ_l, and of their shifts, that remain to be determined. The compatibility conditions between (2.1a) and (2.1b) are given by the lattice zero-curvature condition:

$$\widehat{L}_\kappa M_\kappa = \widetilde{M}_\kappa L_\kappa \ . \tag{2.4}$$

Using the addition formula

$$\Phi_\kappa(x)\Phi_\kappa(y) = \Phi_\kappa(x+y)\left[\zeta(\kappa)+\zeta(x)+\zeta(y)-\zeta(\kappa+x+y)\right] \ , \tag{2.5}$$

the consistency gives rise to

$$\begin{aligned}\sum_{l=1}^{N} \widehat{h}_l k_j \left[\zeta(\widehat{\widehat{\xi}}_i - \widehat{\xi}_l - \alpha) + \zeta(\widehat{\xi}_l - \xi_j - \beta) + \zeta(N\kappa) - \zeta(N\kappa + \widehat{\widetilde{\xi}}_i - \xi_j - \alpha - \beta)\right] = \\ = \quad \sum_{l=1}^{N} \widetilde{k}_l h_j \left[\zeta(\widehat{\widetilde{\xi}}_i - \widetilde{\xi}_l - \beta) + \zeta(\widetilde{\xi}_l - \xi_j - \alpha) + \zeta(N\kappa) - \zeta(N\kappa + \widehat{\widetilde{\xi}}_i - \xi_j - \alpha - \beta)\right] \\ (i,j = 1,\dots,N) \ . \end{aligned} \tag{2.6}$$

Due to the arbitrariness of the spectral parameter κ the equations (2.6) separate into two parts, namely

$$\left(\sum_{l=1}^{N} \widehat{h}_l\right) k_j = \left(\sum_{l=1}^{N} \widetilde{k}_l\right) h_j \quad , \quad (j=1,\dots,N) \ , \tag{2.7a}$$

$$\left\{\sum_{l=1}^{N} \widehat{h}_l \left[\zeta(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_l - \alpha) + \zeta(\widehat{\xi}_l - \xi_j - \beta)\right]\right\} k_j = \left\{\sum_{l=1}^{N} \widetilde{k}_l \left[\zeta(\widehat{\widetilde{\xi}}_i - \widetilde{\xi}_l - \beta) + \zeta(\widetilde{\xi}_l - \xi_j - \alpha)\right]\right\} h_j \\ (i,j=1,\dots,N) \ . \tag{2.7b}$$

Now there are two scenarios which we refer to as the "Landau-Lifschitz type" (or physically, the spin non-zero) case and the "Krichever-Novikov type" (spin zero) cases respectively:

1. Discrete Landau-Lifschitz (LL) type case: $\sum_l h_l \neq 0$, in which case we have that the variables h_j, k_j are proportional to each other, $k_j = \rho h_j$, and after summing (2.7a) we obtain the conservation law:

$$\frac{\sum_{l=1}^{N} \widehat{h}_l}{\sum_{l=1}^{N} h_l} = \frac{\sum_{l=1}^{N} \widetilde{k}_l}{\sum_{l=1}^{N} k_l} \ . \tag{2.8}$$

   and in which case eqs. (2.7b) reduce to:

$$\sum_{l=1}^{N} \left[\zeta(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_l - \alpha)\rho\widehat{h}_l - \zeta(\widehat{\widetilde{\xi}}_i - \widetilde{\xi}_l - \beta)\widetilde{k}_l\right] = \sum_{l=1}^{N} \left[\zeta(\xi_j - \widehat{\xi}_l + \beta)\rho\widehat{h}_l - \zeta(\xi_j - \widetilde{\xi}_l + \alpha)\widetilde{k}_l\right] \ . \\ (i,j=1,\dots,N) \ . \tag{2.9}$$

This system of equations can be reduced under the additional assumption of the conservation law (for the centre of mass):

$$\widetilde{\Xi} + \widehat{\Xi} = \widehat{\widetilde{\Xi}} + \Xi \quad , \quad \Xi \equiv \sum_{l=1}^{N} \xi_l \ . \tag{2.10}$$

2. Krichever-Novikov (KN) type case: $\sum_l h_l = \sum_l k_l = 0$, in which case (2.7a) becomes vacuous. In this case we seek further reductions by the additional constraint $\Xi = \sum_l \xi_l = 0$ (modulo the period lattice of the elliptic functions).

In this paper we will focus primarily on the class of models in # 2, but we will conclude this section by presenting the general structure of the systems that emerge from the Lax system in both cases, and then in the ensuing sections present an alternative analysis for the Lax system of class # 2 for the cases $N = 2$ and $N = 3$.

In order to proceed with the general analysis of (2.9) we use a trick that was employed in [17], based on an elliptic version of the Lagrange interpolation formula (cf. Appendix B) in order to identify the variables h_l, k_l. Consider the following elliptic function, where as a consequence of the conservation law (2.10) for the variables ξ_l the Lagrange interpolation (B.6) of Appendix B is applicable, leading to the following identity:

$$\begin{aligned} F(\xi) &= \prod_{l=1}^{N} \frac{\sigma(\xi - \widehat{\widetilde{\xi}}_l)\sigma(\xi - \xi_l - \alpha - \beta)}{\sigma(\xi - \widehat{\xi}_l - \alpha)\sigma(\xi - \widetilde{\xi}_l - \beta)} \\ &= \sum_{l=1}^{N} \left[\zeta(\xi - \widehat{\xi}_l - \alpha) - \zeta(\eta - \widehat{\xi}_l - \alpha)\right] H_l \\ &\quad + \sum_{l=1}^{N} \left[\zeta(\xi - \widetilde{\xi}_l - \beta) - \zeta(\eta - \widetilde{\xi}_l - \beta)\right] K_l \end{aligned} \tag{2.11}$$

which holds for any four sets of variables ξ_l, $\widehat{\xi}_l$, $\widetilde{\xi}_l$, $\widehat{\widetilde{\xi}}_l$ such that (2.10) holds. In (2.11) η can be any one of the zeroes of $F(\xi)$, i.e. $\widehat{\widetilde{\xi}}_i$ or $\xi_i + \alpha + \beta$, and the coefficients H_j, K_j are given by:

$$H_l = \frac{\prod_{k=1}^{N} \sigma(\widehat{\xi}_l - \widehat{\widetilde{\xi}}_k + \alpha)\sigma(\widehat{\xi}_l - \xi_k - \beta)}{\left[\prod_{k=1}^{N} \sigma(\widehat{\xi}_l - \widetilde{\xi}_k - \gamma)\right] \prod_{k \neq l} \sigma(\widehat{\xi}_l - \widehat{\xi}_k)} \tag{2.12a}$$

$$K_l = \frac{\prod_{k=1}^{N} \sigma(\widetilde{\xi}_l - \widehat{\widetilde{\xi}}_k + \beta)\sigma(\widetilde{\xi}_l - \xi_k - \alpha)}{\left[\prod_{k=1}^{N} \sigma(\widetilde{\xi}_l - \widehat{\xi}_k + \gamma)\right] \prod_{k \neq l} \sigma(\widetilde{\xi}_l - \widetilde{\xi}_k)} \ . \tag{2.12b}$$

Furthermore, the coefficients obey the *identity*:

$$\sum_{l=1}^{N}(H_l+K_l)=0\ . \tag{2.13}$$

Taking $\xi=\widehat{\widetilde{\xi}}_i$, $\eta=\xi_j+\alpha+\beta$ in (2.11) and comparing with (2.7b), we can make the identifications:

$$tH_l=\rho\widehat{h}_l \quad , \quad tK_l=-\widetilde{\rho}\widetilde{h}_l \quad , \quad l=1,\dots,N\ , \tag{2.14}$$

with a function t being an arbitrary proportionality factor. Thus in this case (case 1) by eliminating h_l from (2.14) we get the set of equations

$$\frac{\widetilde{t}}{\widetilde{\rho}}\widetilde{H}_l+\frac{\widehat{t}}{\widehat{\widetilde{\rho}}}\widehat{K}_l=0 \quad , \quad l=1,\dots,N \tag{2.15}$$

which, by inserting the expressions (2.12a) for H_l and K_l, is a system of N equations for $N+2$ unknowns ξ_l, $(l=1,\dots,N)$, and ρ and t. Rewriting this system in explicit form, we obtain the system of N 7-point equations:

$$\prod_{k=1}^{N}\frac{\sigma(\xi_l-\widetilde{\xi}_k+\alpha)\,\sigma(\xi_l-\underset{\frown}{\xi}_k-\beta)\,\sigma(\xi_l-\underset{\smile}{\widehat{\xi}}_k+\gamma)}{\sigma(\xi_l-\widehat{\xi}_k+\beta)\,\sigma(\xi_l-\underset{\smile}{\xi}_k-\alpha)\,\sigma(\xi_l-\underset{\frown}{\widetilde{\xi}}_k-\gamma)}=-p \tag{2.16}$$

for $N+1$ variables ξ_i $(i=1,\dots,N)$ and $p=\underset{\smile}{t}\,\underset{\frown}{\rho}/(\underset{\frown}{t}\,\rho)$, supplemented with (2.10) which fixes the discrete dynamics of the centre of mass Ξ . In (2.16) the under-accents $\underset{\smile}{\cdot}$ and $\underset{\frown}{\cdot}$ denote reverse lattice shifts, i.e., $\underset{\smile}{\xi}_i(n,m)=\xi_i(n-1,m)$ and $\underset{\frown}{\xi}_i(n,m)=\xi_i(n,m-1)$ respectively. These equations and their rational forms will be investigated more in detail in a future publication. We mention here only that the one-step periodic reduction, $\widetilde{\chi}_\kappa=\lambda\chi_\kappa$, in this case leads to an implicit system of ordinary difference equations which amounts to a the time-discretization of the Ruijsenaars (relativistic Calogero-Moser) model, cf. [17]. In the remainder of the paper we will concentrate on the case #2 which constitutes higher rank analogues of Adler's lattice equation in 3-leg form, and we will perform a different kind of analysis in that case.

3 Elliptic Lax pairs for 3-leg lattice systems

In this section we will focus on case #2 of general elliptic Lax systems introduced in the previous section, corresponding to the "spin-zero" case (where $\sum_{l=1}^{N}h_l=\sum_{l=1}^{N}k_l=$

0). We will first demonstrate in the case $N = 2$ of this system how the 3-leg form of Adler's equation arises in a natural way from this Lax pair. In fact, it turns out that the elaboration of the compatibility conditions for this Lax pair immediately produces the required equations, and is far less laborious than of the consistency-around-the-cube (CAC) Lax pair of [16] yielding the corresponding rational form of Q4. Next we will analyse the much more generic case of $N = 3$, and produce a novel system of elliptic lattice equations, which constitutes the main result of this paper. We also present the structure of the lattice system arising form the scheme for general N, based on similar ingredients as the ones used in the case #1 elaborated in the previous section, but subject to slightly different conditions.

3.1 Case N=2: Elliptic Lax Pair for the Adler 3-leg lattice equation

Let $\xi = \xi_{n,m}$ be a function of the discrete independent variables n, m for which we want to derive a lattice equation from the following Lax pair:

$$\widetilde{\chi} = L_\kappa \chi = \lambda \begin{pmatrix} \Phi_{2\kappa}(\widetilde{\xi} - \xi - \alpha) & -\Phi_{2\kappa}(\widetilde{\xi} + \xi - \alpha) \\ \Phi_{2\kappa}(-\widetilde{\xi} - \xi - \alpha) & -\Phi_{2\kappa}(-\widetilde{\xi} + \xi - \alpha) \end{pmatrix} \chi \tag{3.1a}$$

$$\widehat{\chi} = M_\kappa \chi = \mu \begin{pmatrix} \Phi_{2\kappa}(\widehat{\xi} - \xi - \beta) & -\Phi_{2\kappa}(\widehat{\xi} + \xi - \beta) \\ \Phi_{2\kappa}(-\widehat{\xi} - \xi - \beta) & -\Phi_{2\kappa}(-\widehat{\xi} + \xi - \beta) \end{pmatrix} \chi \ , \tag{3.1b}$$

in which the coefficients λ and μ are functions $\lambda = \lambda(\xi, \widetilde{\xi}; \alpha)$ and $\mu = \mu(\xi, \widehat{\xi}; \beta)$, respectively. The explicit form of which will be derived subsequently, but these forms will actually not be relevant for the determination of the resulting lattice equation, which is Adler's system in 3-leg form. The discrete zero-curvature condition (2.4) can, once again, be analysed using the addition formula (2.5) for the Lamé function Φ_κ and analyzed entry-by-entry. Applying this to each entry of both the left-hand side and right-hand side of (2.4) we observe that in all four entries a common factor containing the spectral parameter

κ will drop out and that we are left with the following four relations:

$$
\begin{aligned}
&\widehat{\lambda}\mu \left[\zeta(\widehat{\widetilde{\xi}} - \widehat{\xi} - \alpha) + \zeta(\widehat{\xi} - \xi - \beta) - \zeta(\widehat{\widetilde{\xi}} + \widehat{\xi} - \alpha) + \zeta(\widehat{\xi} + \xi + \beta)\right] \\
&\quad = \widetilde{\mu}\lambda \left[\zeta(\widehat{\widetilde{\xi}} - \widetilde{\xi} - \beta) + \zeta(\widetilde{\xi} - \xi - \alpha) - \zeta(\widehat{\widetilde{\xi}} + \widetilde{\xi} - \beta) + \zeta(\widetilde{\xi} + \xi + \alpha)\right] \qquad (3.2a) \\
&\widehat{\lambda}\mu \left[\zeta(\widehat{\widetilde{\xi}} - \widehat{\xi} - \alpha) + \zeta(\widehat{\xi} + \xi - \beta) - \zeta(\widehat{\widetilde{\xi}} + \widehat{\xi} - \alpha) + \zeta(\widehat{\xi} - \xi + \beta)\right] \\
&\quad = \widetilde{\mu}\lambda \left[\zeta(\widehat{\widetilde{\xi}} - \widetilde{\xi} - \beta) + \zeta(\widetilde{\xi} + \xi - \alpha) - \zeta(\widehat{\widetilde{\xi}} + \widetilde{\xi} - \beta) + \zeta(\widetilde{\xi} + \xi - \alpha)\right] \qquad (3.2b) \\
&\widehat{\lambda}\mu \left[\zeta(-\widehat{\widetilde{\xi}} - \widehat{\xi} - \alpha) + \zeta(\widehat{\xi} - \xi - \beta) - \zeta(-\widehat{\widetilde{\xi}} + \widehat{\xi} - \alpha) + \zeta(\widehat{\xi} + \xi + \beta)\right] \\
&\quad = \widetilde{\mu}\lambda \left[\zeta(\widehat{\widetilde{\xi}} - \widetilde{\xi} - \beta) + \zeta(\widetilde{\xi} - \xi - \alpha) - \zeta(\widehat{\widetilde{\xi}} + \widetilde{\xi} - \beta) + \zeta(\widetilde{\xi} + \xi + \alpha)\right] \qquad (3.2c) \\
&\widehat{\lambda}\mu \left[\zeta(-\widehat{\widetilde{\xi}} - \widehat{\xi} - \alpha) + \zeta(\widehat{\xi} + \xi - \beta) - \zeta(-\widehat{\widetilde{\xi}} + \widehat{\xi} - \alpha) + \zeta(\widehat{\xi} - \xi + \beta)\right] \\
&\quad = \widetilde{\mu}\lambda \left[\zeta(-\widehat{\widetilde{\xi}} - \widetilde{\xi} - \beta) + \zeta(\widetilde{\xi} + \xi - \alpha) - \zeta(-\widehat{\widetilde{\xi}} + \widetilde{\xi} - \beta) + \zeta(\widetilde{\xi} - \xi + \alpha)\right] \qquad (3.2d)
\end{aligned}
$$

Using the identity (2.5) these four relations can be rewritten as:

$$
\begin{aligned}
&\widehat{\lambda}\mu \frac{\sigma(2\widehat{\xi})\,\sigma(\widehat{\widetilde{\xi}} + \xi + \beta - \alpha)}{\sigma(\widehat{\widetilde{\xi}} - \widehat{\xi} - \alpha)\,\sigma(\widehat{\widetilde{\xi}} + \widehat{\xi} - \alpha)\,\sigma(\widehat{\xi} - \xi - \beta)\,\sigma(\widehat{\xi} + \xi + \beta)} \\
&\quad = \widetilde{\mu}\lambda \frac{\sigma(2\widetilde{\xi})\,\sigma(\widehat{\widetilde{\xi}} + \xi + \alpha - \beta)}{\sigma(\widehat{\widetilde{\xi}} - \widetilde{\xi} - \beta)\,\sigma(\widehat{\widetilde{\xi}} + \widetilde{\xi} - \beta)\,\sigma(\widetilde{\xi} - \xi - \alpha)\,\sigma(\widetilde{\xi} + \xi + \alpha)} \qquad (3.3a) \\
&\widehat{\lambda}\mu \frac{\sigma(2\widehat{\xi})\,\sigma(\widehat{\widetilde{\xi}} - \xi + \beta - \alpha)}{\sigma(\widehat{\widetilde{\xi}} - \widehat{\xi} - \alpha)\,\sigma(\widehat{\widetilde{\xi}} + \widehat{\xi} - \alpha)\,\sigma(\widehat{\xi} - \xi + \beta)\,\sigma(\widehat{\xi} + \xi - \beta)} \\
&\quad = \widetilde{\mu}\lambda \frac{\sigma(2\widetilde{\xi})\,\sigma(\widehat{\widetilde{\xi}} - \xi + \alpha - \beta)}{\sigma(\widehat{\widetilde{\xi}} - \widetilde{\xi} - \beta)\,\sigma(\widehat{\widetilde{\xi}} + \widetilde{\xi} - \beta)\,\sigma(\widetilde{\xi} - \xi + \alpha)\,\sigma(\widetilde{\xi} + \xi - \alpha)} \qquad (3.3b) \\
&\widehat{\lambda}\mu \frac{\sigma(2\widehat{\xi})\,\sigma(\widehat{\widetilde{\xi}} - \xi - \beta + \alpha)}{\sigma(\widehat{\widetilde{\xi}} - \widehat{\xi} + \alpha)\,\sigma(\widehat{\widetilde{\xi}} + \widehat{\xi} + \alpha)\,\sigma(\widehat{\xi} - \xi - \beta)\,\sigma(\widehat{\xi} + \xi + \beta)} \\
&\quad = \widetilde{\mu}\lambda \frac{\sigma(2\widetilde{\xi})\,\sigma(\widehat{\widetilde{\xi}} - \xi - \alpha + \beta)}{\sigma(\widehat{\widetilde{\xi}} - \widetilde{\xi} + \beta)\,\sigma(\widehat{\widetilde{\xi}} + \widetilde{\xi} + \beta)\,\sigma(\widetilde{\xi} - \xi - \alpha)\,\sigma(\widetilde{\xi} + \xi + \alpha)} \qquad (3.3c) \\
&\widehat{\lambda}\mu \frac{\sigma(2\widehat{\xi})\,\sigma(\widehat{\widetilde{\xi}} + \xi - \beta + \alpha)}{\sigma(\widehat{\widetilde{\xi}} - \widehat{\xi} + \alpha)\,\sigma(\widehat{\widetilde{\xi}} + \widehat{\xi} + \alpha)\,\sigma(\widehat{\xi} - \xi + \beta)\,\sigma(\widehat{\xi} + \xi - \beta)} \\
&\quad = \widetilde{\mu}\lambda \frac{\sigma(2\widetilde{\xi})\,\sigma(\widehat{\widetilde{\xi}} + \xi - \alpha + \beta)}{\sigma(\widehat{\widetilde{\xi}} - \widetilde{\xi} + \beta)\,\sigma(\widehat{\widetilde{\xi}} + \widetilde{\xi} + \beta)\,\sigma(\widetilde{\xi} - \xi + \alpha)\,\sigma(\widetilde{\xi} + \xi - \alpha)}. \qquad (3.3d)
\end{aligned}
$$

Eliminating λ and μ, simply by dividing pairwise the relations over each other, we obtain directly the 3-leg formulae. In fact, we obtain two seemingly different-looking equations for ξ, namely:

$$\frac{\sigma(\widetilde{\xi}-\xi+\alpha)\,\sigma(\widetilde{\xi}+\xi-\alpha)}{\sigma(\widetilde{\xi}-\xi-\alpha)\,\sigma(\widetilde{\xi}+\xi+\alpha)}\,\frac{\sigma(\widehat{\xi}-\xi-\beta)\,\sigma(\widehat{\xi}+\xi+\beta)}{\sigma(\widehat{\xi}-\xi+\beta)\,\sigma(\widehat{\xi}+\xi-\beta)}=\frac{\sigma(\widehat{\widetilde{\xi}}-\xi-\gamma)\,\sigma(\widehat{\widetilde{\xi}}+\xi+\gamma)}{\sigma(\widehat{\widetilde{\xi}}-\xi+\gamma)\,\sigma(\widehat{\widetilde{\xi}}+\xi-\gamma)}\ , \tag{3.4a}$$

in which as before $\gamma=\beta-\alpha$, and

$$\frac{\sigma(\widehat{\widetilde{\xi}}-\widehat{\xi}+\alpha)\,\sigma(\widehat{\widetilde{\xi}}+\widehat{\xi}+\alpha)}{\sigma(\widehat{\widetilde{\xi}}-\widehat{\xi}-\alpha)\,\sigma(\widehat{\widetilde{\xi}}+\widehat{\xi}-\alpha)}\,\frac{\sigma(\widehat{\widetilde{\xi}}-\widetilde{\xi}-\beta)\,\sigma(\widehat{\widetilde{\xi}}+\widetilde{\xi}-\beta)}{\sigma(\widehat{\widetilde{\xi}}-\widetilde{\xi}+\beta)\,\sigma(\widehat{\widetilde{\xi}}+\widetilde{\xi}+\beta)}=\frac{\sigma(\widehat{\widetilde{\xi}}-\xi-\gamma)\,\sigma(\widehat{\widetilde{\xi}}+\xi-\gamma)}{\sigma(\widehat{\widetilde{\xi}}-\xi+\gamma)\,\sigma(\widehat{\widetilde{\xi}}+\xi+\gamma)}\ , \tag{3.4b}$$

but actually these two equations are equivalent. The first equation (3.4a) is identical to (1.6), namely the 3-leg form of the Adler lattice equation.The second equation (3.4b) is obtained from the first by interchanging $\xi\leftrightarrow\widehat{\widetilde{\xi}}$, $\alpha\leftrightarrow\beta$, which is a symmetry of the equation. The equivalence between these two forms is made manifest by passing to the rational form (1.2) of the equation, and the latter connection can be seen to be a consequence of an interesting identity given in the following statement.

Proposition 3.1. *For arbitrary (complex) variables X, Y, and Z, we have the following identity*

$$\begin{aligned}
&(X-\wp(\xi+\alpha))(Y-\wp(\xi-\beta))(Z-\wp(\xi-\alpha+\beta))\\
&\quad -t^2(X-\wp(\xi-\alpha))(Y-\wp(\xi+\beta))(Z-\wp(\xi+\alpha-\beta))\\
&\quad = s\big[(-aB-bA)(\wp(\xi)(XY+YZ+XZ)+XYZ)+(b^2A+a^2B)(Z\wp(\xi)+XY)\\
&\quad +((b^2A+a^2B)-B(a-b)(a-c))(X\wp(\xi)+YZ)+(-A(b-a)(b-c)+(b^2A+a^2B))\\
&\quad \times(Y\wp(\xi)+XZ)+(aB(a-b)(a-c)+bA(b-a)(b-c)-Ab^3-Ba^3)(\wp(\xi)+X+Y+Z)\\
&\quad +A(b^2-(a-b)(c-b)))^2+B(a^2-(b-a)(c-a))^2-ABC(a-b)+(A+B)XYZ\wp(\xi)\big],
\end{aligned} \tag{3.5}$$

in which

$$t=\frac{\sigma(\xi-\alpha)\sigma(\xi+\beta)\sigma(\xi+\alpha-\beta)}{\sigma(\xi+\alpha)\sigma(\xi-\beta)\sigma(\xi-\alpha+\beta)},\quad s=\frac{1-t^2}{(A+B)\wp(\xi)-Ab-aB}. \tag{3.6}$$

A (computational) proof of the Proposition 3.1 is given in Appendix A. Identifying $u = \wp(\xi)$, $X = \widetilde{u} = \wp(\widetilde{\xi})$, $Y = \widehat{u} = \wp(\widehat{\xi})$ and $Z = \widehat{\widetilde{u}} = \wp(\widehat{\widetilde{\xi}})$, and using

$$\wp(\xi) - \wp(\eta) = \frac{\sigma(\eta+\xi)\,\sigma(\eta-\xi)}{\sigma^2(\eta)\sigma^2(\xi)} \quad , \tag{3.7}$$

it is not hard to see that the elliptic identity (3.5) relates the rational form of Adler's equation in the Weierstrass case (1.2) and 3-leg (3.4a). Since the Adler system (1.2) is manifestly invariant under the replacements $u \leftrightarrow \widehat{\widetilde{u}}$, $\alpha \leftrightarrow \beta$ – whilst *not* interchanging $\widetilde{u}$ and $\widehat{u}$ – (this being a particular aspect of the D_4-symmetry of the equation), the 3-leg form (3.4a) is also invariant under the parallel exchange on the level of the uniformising variables: $\xi \leftrightarrow \widehat{\widetilde{\xi}}$, $\alpha \leftrightarrow \beta$. This is the symmetry that connects the two forms (3.4a) and (3.4b), which are hence equivalent.

Remark 1: The coefficients λ and μ are determined by the condition that the dynamical equation for the determinants of the Lax matrices L_κ, M_κ need to be trivially satisfied. Thus a possible choice for λ and μ is to determine these factors such that $\det(L_\kappa)$ and $\det(M_\kappa)$ are proportional to constants (i.e. independent of ξ), which leads to the following expressions

$$\lambda = \left(\frac{H(u,\widetilde{u},a)}{AU\widetilde{U}}\right)^{1/2} \quad , \quad \mu = \left(\frac{H(u,\widehat{u},b)}{BU\widehat{U}}\right)^{1/2} \quad , \tag{3.8}$$

where $u = \wp(\xi)$, $U = r(u) = \wp'(\xi)$, and similary $\widetilde{u} = \wp(\widetilde{\xi})$, $\widetilde{U} = r(\widetilde{u}) = \wp'(\widetilde{\xi})$, and $\widehat{u} = \wp(\widehat{\xi})$, $\widehat{U} = r(\widehat{u}) = \wp'(\widehat{\xi})$. The symmetric triquadratic function H is given by

$$H(u,v,a) \equiv \left(uv + au + av + \frac{g_2}{4}\right)^2 - (4auv - g_3)(u+v+a) \ , \tag{3.9}$$

and which can be obtained in the following form in terms of σ-function

$$\begin{aligned} H(u,v,a) &= (u-v)^2\left[\frac{1}{4}\left(\frac{U-V}{u-v}\right)^2 - (u+v+a)\right]\left[\frac{1}{4}\left(\frac{U+V}{u-v}\right)^2 - (u+v+a)\right] \\ &= \frac{\sigma(\xi+\eta+\alpha)\,\sigma(\xi+\eta-\alpha)\,\sigma(\xi-\eta+\alpha)\,\sigma(\xi-\eta-\alpha)}{\sigma^4(\xi)\,\sigma^4(\eta)\,\sigma^4(\alpha)} \ , \end{aligned} \tag{3.10}$$

in which $U^2 \equiv r(u)$, $V^2 \equiv r(v)$. We also have the expression in terms of the polynomial of the curve:

$$\big[r(u) + r(a) - 4(u-a)^2(u+v+a)\big]^2 - 4r(u)\,r(a) = 16(u-a)^2 H(u,v,a) \ . \tag{3.11}$$

We further note at this point that the discriminant of the triquadratic in each argument factorises:

$$H_v^2 - 2H\, H_{vv} = r(a)r(u) \ . \tag{3.12}$$

In [4] the discriminant properties of affine-linear quadrilaterals and their relation with the corresponding biquadratics and their discriminants, were exploited to tighten the classification result of [3].

Remark 2: An alternative derivation of the $N=2$ case can be given by using the system of equations (2.9). In this case the variables H_l and K_l take on the following forms, setting $\xi_1 = -\xi_2 = \xi$:

$$\begin{aligned}
H_1 &= \frac{\sigma(\widehat{\xi} - \widehat{\widetilde{\xi}} + \alpha)\,\sigma(\widehat{\xi} + \widehat{\widetilde{\xi}} + \alpha)\,\sigma(\widehat{\xi} - \xi - \beta)\,\sigma(\widehat{\xi} + \xi - \beta)}{\sigma(\widehat{\xi} - \widetilde{\xi} - \gamma)\,\sigma(\widehat{\xi} + \widetilde{\xi} - \gamma)\,\sigma(2\widehat{\xi})} \ , && (3.13a) \\
H_2 &= \frac{\sigma(-\widehat{\xi} - \widehat{\widetilde{\xi}} + \alpha)\,\sigma(-\widehat{\xi} + \widehat{\widetilde{\xi}} + \alpha)\,\sigma(-\widehat{\xi} - \xi - \beta)\,\sigma(-\widehat{\xi} + \xi - \beta)}{\sigma(-\widehat{\xi} - \widetilde{\xi} - \gamma)\,\sigma(-\widehat{\xi} + \widetilde{\xi} - \gamma)\,\sigma(-2\widehat{\xi})} \ , && (3.13b) \\
K_1 &= \frac{\sigma(\widetilde{\xi} - \widehat{\widetilde{\xi}} + \beta)\,\sigma(\widetilde{\xi} + \widehat{\widetilde{\xi}} + \beta)\,\sigma(\widetilde{\xi} - \xi - \alpha)\,\sigma(\widetilde{\xi} + \xi - \alpha)}{\sigma(\widetilde{\xi} - \widehat{\xi} + \gamma)\,\sigma(\widetilde{\xi} + \widehat{\xi} + \gamma)\,\sigma(2\widetilde{\xi})} \ , && (3.13c) \\
K_2 &= \frac{\sigma(-\widetilde{\xi} - \widehat{\widetilde{\xi}} + \beta)\,\sigma(-\widetilde{\xi} + \widehat{\widetilde{\xi}} + \beta)\,\sigma(-\widetilde{\xi} - \xi - \alpha)\,\sigma(-\widetilde{\xi} + \xi - \alpha)}{\sigma(-\widetilde{\xi} - \widehat{\xi} + \gamma)\,\sigma(-\widetilde{\xi} + \widehat{\xi} + \gamma)\,\sigma(-2\widetilde{\xi})} \ , && (3.13d)
\end{aligned}$$

The identity $H_1 + H_2 = 0$ upon inserting the above expressions yield the equation:

$$\left[\frac{\sigma(\widetilde{\xi} + \xi + \alpha)\,\sigma(\widetilde{\xi} - \xi - \alpha)}{\sigma(\widetilde{\xi} + \xi - \alpha)\,\sigma(\widetilde{\xi} - \xi + \alpha)}\right]^{\widehat{}} \frac{\sigma(\widehat{\xi} + \xi - \beta)\,\sigma(\widehat{\xi} - \xi - \beta)}{\sigma(\widehat{\xi} + \xi + \beta)\,\sigma(\widehat{\xi} - \xi + \beta)} = \frac{\sigma(\widetilde{\xi} + \widehat{\xi} - \gamma)\,\sigma(\widetilde{\xi} - \widehat{\xi} + \gamma)}{\sigma(\widetilde{\xi} - \widehat{\xi} - \gamma)\,\sigma(\widetilde{\xi} + \widehat{\xi} + \gamma)} \ , \tag{3.14}$$

which is equivalent to the elliptic lattice system (1.2) under the same changes of variables as discussed before. In fact, (3.14) can be obtained from (3.4a) by interchanging: $\xi \leftrightarrow \widehat{\xi}$ and $\widehat{\widetilde{\xi}} \leftrightarrow \widetilde{\xi}$. Similarly, the identity $K_1 + K_2 = 0$ upon inserting the expressions (3.13c) and (3.13d) for K_1 and K_2 yields a similar equation to (3.14) which can be obtained from (3.4a) by interchanging: $\xi \leftrightarrow \widetilde{\xi}$ and $\widehat{\widetilde{\xi}} \leftrightarrow \widehat{\xi}$. Thus, we recover from the scheme proposed in the previous section the Adler system in the various 3-leg forms based at different vertices of the elementary quadrilateral.

3.2 Case N=3:

To generalise the results in the previous subsection to the rank 3 case, we consider the following form of a Lax representation on the lattice:

$$\widetilde{\chi} = \begin{pmatrix} h_1\Phi_{3\kappa}(\widetilde{\xi}_1-\xi_1-\alpha) & h_2\Phi_{3\kappa}(\widetilde{\xi}_1-\xi_2-\alpha) & h_3\Phi_{3\kappa}(\widetilde{\xi}_1-\xi_3-\alpha) \\ h_1\Phi_{3\kappa}(\widetilde{\xi}_2-\xi_1-\alpha) & h_2\Phi_{3\kappa}(\widetilde{\xi}_2-\xi_2-\alpha) & h_3\Phi_{3\kappa}(\widetilde{\xi}_2-\xi_3-\alpha) \\ h_1\Phi_{3\kappa}(\widetilde{\xi}_3-\xi_1-\alpha) & h_2\Phi_{3\kappa}(\widetilde{\xi}_3-\xi_2-\alpha) & h_3\Phi_{3\kappa}(\widetilde{\xi}_3-\xi_3-\alpha) \end{pmatrix}\chi\ , \tag{3.15a}$$

$$\widehat{\chi} = \begin{pmatrix} k_1\Phi_{3\kappa}(\widehat{\xi}_1-\xi_1-\beta) & k_2\Phi_{3\kappa}(\widehat{\xi}_1-\xi_2-\beta) & k_3\Phi_{3\kappa}(\widehat{\xi}_1-\xi_3-\beta) \\ k_1\Phi_{3\kappa}(\widehat{\xi}_2-\xi_1-\beta) & k_2\Phi_{3\kappa}(\widehat{\xi}_2-\xi_2-\beta) & k_3\Phi_{3\kappa}(\widehat{\xi}_2-\xi_3-\beta) \\ k_1\Phi_{3\kappa}(\widehat{\xi}_3-\xi_1-\beta) & k_2\Phi_{3\kappa}(\widehat{\xi}_3-\xi_2-\beta) & k_3\Phi_{3\kappa}(\widehat{\xi}_3-\xi_3-\beta) \end{pmatrix}\chi\ , \tag{3.15b}$$

subject to $\sum_{i=1}^3 h_i = \sum_{i=1}^3 k_i = 0$, and where the coefficients h_j, k_j are some functions of the variables ξ_j, and of their shifts. The compatibility conditions (2.4) of this Lax pair results in a coupled set of Lax equations in terms of the three variables ξ_j as we shall demonstrate by performing a similar type of analysis as in the case $N = 2$, which in this case is understandably more involved.

Eliminating[2], $h_3 = -h_1 - h_2$ and $k_3 = -k_1 - k_2$ we obtain from (2.7b) the following system of equations:

$$\begin{aligned} &\sum_{l=1}^{2}\widehat{h}_l k_j\left[\zeta(\widehat{\widetilde{\xi}}_i-\widehat{\xi}_l-\alpha)+\zeta(\widehat{\xi}_l-\xi_j-\beta)-\zeta(\widehat{\widetilde{\xi}}_i-\widehat{\xi}_3-\alpha)-\zeta(\widehat{\xi}_3-\xi_j-\beta)\right] \\ &=\sum_{l=1}^{2}\widetilde{k}_l h_j\left[\zeta(\widehat{\widetilde{\xi}}_i-\widetilde{\xi}_l-\beta)+\zeta(\widetilde{\xi}_l-\xi_j-\alpha)-\zeta(\widehat{\widetilde{\xi}}_i-\widetilde{\xi}_3-\beta)-\zeta(\widetilde{\xi}_3-\xi_j-\alpha)\right] \\ &\forall\ i,j=1,2,3. \end{aligned} \tag{3.16}$$

and using the addition formula (2.5) we next get:

$$\begin{aligned} &\sum_{l=1}^{2}\widehat{h}_l k_j\frac{\sigma(\widehat{\widetilde{\xi}}_i-\widehat{\xi}_l-\widehat{\xi}_3+\xi_j-\alpha+\beta)\sigma(\widehat{\xi}_l-\widehat{\xi}_3)}{\sigma(\widehat{\widetilde{\xi}}_i-\widehat{\xi}_l-\alpha)\sigma(\widehat{\xi}_l-\xi_j-\beta)\sigma(\widehat{\widetilde{\xi}}_i-\widehat{\xi}_3-\alpha)\sigma(\widehat{\xi}_3-\xi_j-\beta)}= \\ &\qquad=\sum_{l=1}^{2}\widetilde{k}_l h_j\frac{\sigma(\widehat{\widetilde{\xi}}_i-\widetilde{\xi}_l-\widetilde{\xi}_3+\xi_j+\alpha-\beta)\sigma(\widetilde{\xi}_l-\widetilde{\xi}_3)}{\sigma(\widehat{\widetilde{\xi}}_i-\widetilde{\xi}_l-\beta)\sigma(\widetilde{\xi}_l-\xi_j-\alpha)\sigma(\widehat{\widetilde{\xi}}_i-\widetilde{\xi}_3-\beta)\sigma(\widetilde{\xi}_3-\xi_j-\alpha)} \\ &\forall\ i,j=1,2,3. \end{aligned} \tag{3.17}$$

[2] Equivalently, we could have eliminated h_1 or h_2 and k_1 or k_2 yielding equivalent results.

To write (3.17) in a more concise way, we denote the coefficients on the l.h.s. and r.h.s. of the equation as $A_{ilj} \equiv A_{ilj}(\widehat{\widetilde{\xi}}, \widehat{\xi}, \xi; \alpha, \beta)$ and $B_{ilj} \equiv B_{ilj}(\widehat{\widetilde{\xi}}, \widetilde{\xi}, \xi; \alpha, \beta)$ respectively. Noting the common factors h_j/k_j $(j = 1, 2, 3)$ in these equations, we next derive the system of six equations

$$\frac{h_j}{k_j} = \frac{A_{11j}\widehat{h}_1 + A_{12j}\widehat{h}_2}{B_{11j}\widetilde{k}_1 + B_{12j}\widetilde{k}_2} = \frac{A_{21j}\widehat{h}_1 + A_{22j}\widehat{h}_2}{B_{21j}\widetilde{k}_1 + B_{22j}\widetilde{k}_2} = \frac{A_{31j}\widehat{h}_1 + A_{32j}\widehat{h}_2}{B_{31j}\widetilde{k}_1 + B_{32j}\widetilde{k}_2} \quad (j = 1, 2, 3) \ . \tag{3.18}$$

We can rewrite the resulting set of relation (3.18) as

$$\begin{aligned}
&(A_{11j}B_{21j} - A_{21j}B_{11j})\widehat{h}_1\widetilde{k}_1 + (A_{11j}B_{22j} - A_{21j}B_{12j})\widehat{h}_1\widetilde{k}_2 \\
&\qquad + (A_{12j}B_{21j} - A_{22j}B_{11j})\widehat{h}_2\widetilde{k}_1 + (A_{12j}B_{22j} - A_{22j}B_{12j})\widehat{h}_2\widetilde{k}_2 = 0 \\
&(A_{11j}B_{31j} - A_{31j}B_{11j})\widehat{h}_1\widetilde{k}_1 + (A_{11j}B_{32j} - A_{31j}B_{12j})\widehat{h}_1\widetilde{k}_2 \\
&\qquad + (A_{12j}B_{31j} - A_{32j}B_{11j})\widehat{h}_2\widetilde{k}_1 + (A_{12j}B_{32j} - A_{32j}B_{12j})\widehat{h}_2\widetilde{k}_2 = 0 \\
&(A_{21j}B_{31j} - A_{31j}B_{21j})\widehat{h}_1\widetilde{k}_1 + (A_{21j}B_{32j} - A_{31j}B_{22j})\widehat{h}_1\widetilde{k}_2 \\
&\qquad + (A_{22j}B_{31j} - A_{32j}B_{21j})\widehat{h}_2\widetilde{k}_1 + (A_{22j}B_{32j} - A_{32j}B_{22j})\widehat{h}_2\widetilde{k}_2 = 0 \\
&\qquad (j = 1, 2, 3) \ ,
\end{aligned} \tag{3.19}$$

where

$$A_{ilj} = \frac{\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_l - \widehat{\xi}_3 + \xi_j - \alpha + \beta)\,\sigma(\widehat{\xi}_l - \widehat{\xi}_3)}{\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_l - \alpha)\,\sigma(\widehat{\xi}_l - \xi_j - \beta)\,\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_3 - \alpha)\,\sigma(\widehat{\xi}_3 - \xi_j - \beta)} \ , \tag{3.20a}$$

$$B_{ilj} = \frac{\sigma(\widehat{\widetilde{\xi}}_i - \widetilde{\xi}_l - \widetilde{\xi}_3 + \xi_j + \alpha - \beta)\,\sigma(\widetilde{\xi}_l - \widetilde{\xi}_3)}{\sigma(\widehat{\widetilde{\xi}}_i - \widetilde{\xi}_l - \beta)\,\sigma(\widetilde{\xi}_l - \xi_j - \alpha)\,\sigma(\widehat{\widetilde{\xi}}_i - \widetilde{\xi}_3 - \beta)\,\sigma(\widetilde{\xi}_3 - \xi_j - \alpha)} \ . \tag{3.20b}$$

We observe that these homogeneous bilinear systems for the variables $\widehat{h}_1$, $\widetilde{k}_1$, $\widehat{h}_2$ and $\widetilde{k}_2$ can be resolved by using Cayley's 3-dimensional $2 \times 2 \times 2$-hyperdeterminant [9]. Let us recall the general statement (cf. also [12]):

Definition 3.1. *The hyperdeterminant of* $2 \times 2 \times 2$ *hyper-matrix* $A = (a_{ijk})$ $(i, j, k = 0, 1)$

is given by:

$$\mathrm{Det}(A) = \left[\det\begin{pmatrix} a_{000} & a_{001} \\ a_{110} & a_{111} \end{pmatrix} + \det\begin{pmatrix} a_{100} & a_{010} \\ a_{101} & a_{011} \end{pmatrix}\right]^2 - 4\det\begin{pmatrix} a_{000} & a_{001} \\ a_{010} & a_{011} \end{pmatrix}\det\begin{pmatrix} a_{100} & a_{101} \\ a_{110} & a_{111} \end{pmatrix}. \quad (3.21)$$

Its main property is the following:

Proposition 3.2. *The hyper-determinant* (3.21) *vanishes identically iff the following set of bilinear equations with six unknowns*

$$\begin{aligned}
&a_{001}x_0y_0 + a_{011}x_0y_1 + a_{101}x_1y_0 + a_{111}x_1y_1 = 0,\\
&a_{000}x_0y_0 + a_{010}x_0y_1 + a_{100}x_1y_0 + a_{110}x_1y_1 = 0,\\
&a_{010}x_0z_0 + a_{011}x_0z_1 + a_{110}x_1z_0 + a_{111}x_1z_1 = 0,\\
&a_{000}x_0z_0 + a_{001}x_0z_1 + a_{100}x_1z_0 + a_{101}x_1z_1 = 0,\\
&a_{100}y_0z_0 + a_{101}y_0z_1 + a_{110}y_1z_0 + a_{111}y_1z_1 = 0,\\
&a_{000}y_0z_0 + a_{001}x_0z_1 + a_{010}y_1z_0 + a_{011}y_1z_1 = 0,
\end{aligned} \quad (3.22)$$

has a non-trivial solution (i.e., for which none of the vectors $\boldsymbol{x} = (x_0, x_1)$*,* $\boldsymbol{y} = (y_0, y_1)$*,* $\boldsymbol{z} = (z_0, z_1)$ *are equal to the zero vector).*

A proof of this statement can be found in [21]. The cubic hyper-matrix A can be illustrated by the following diagram of entries

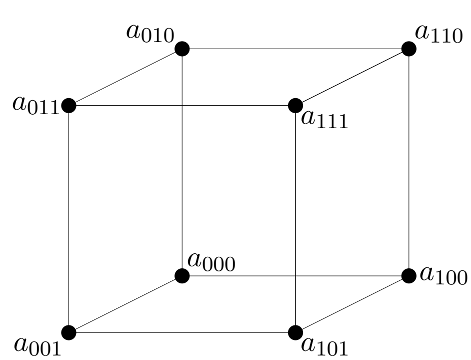

Figure 3: Cayley Cube

In the case at hand, the components a_{ijk} can be readily identified by comparing (3.19) with the system (3.22) and the variables x_i, y_j with the $\widehat{h}_i$ and $\widetilde{k}_j$ respectively. Noting that these particular coefficients are all 2×2 determinants, it turns out that the following *compound theorem for hyper-determinants* is directly applicable.

Lemma 3.1 (Compound Theorem for 2×2×2 hyper-determinants)**.** *The following identity holds for the compound hyper- determinants of format* $2 \times 2 \times 2$*:*

$$\left(\begin{vmatrix} \begin{vmatrix} a & a'' \\ b & b'' \end{vmatrix} & \begin{vmatrix} a' & a'' \\ d' & d'' \end{vmatrix} \\ \begin{vmatrix} c & c'' \\ b & b'' \end{vmatrix} & \begin{vmatrix} c' & c'' \\ d' & d'' \end{vmatrix} \end{vmatrix} + \begin{vmatrix} \begin{vmatrix} a' & a'' \\ b' & b'' \end{vmatrix} & \begin{vmatrix} a & a'' \\ d & d'' \end{vmatrix} \\ \begin{vmatrix} c' & c'' \\ b' & b'' \end{vmatrix} & \begin{vmatrix} c & c'' \\ d & d'' \end{vmatrix} \end{vmatrix} \right)^2$$

$$-4 \begin{vmatrix} \begin{vmatrix} a & a'' \\ b & b'' \end{vmatrix} & \begin{vmatrix} a & a'' \\ d & d'' \end{vmatrix} \\ \begin{vmatrix} c & c'' \\ b & b'' \end{vmatrix} & \begin{vmatrix} c & c'' \\ d & d'' \end{vmatrix} \end{vmatrix} \cdot \begin{vmatrix} \begin{vmatrix} a' & a'' \\ b' & b'' \end{vmatrix} & \begin{vmatrix} a' & a'' \\ d' & d'' \end{vmatrix} \\ \begin{vmatrix} c' & c'' \\ b' & b'' \end{vmatrix} & \begin{vmatrix} c' & c'' \\ d' & d'' \end{vmatrix} \end{vmatrix} = \begin{vmatrix} \begin{vmatrix} a & a'' \\ c & c'' \end{vmatrix} & \begin{vmatrix} b & b'' \\ d & d'' \end{vmatrix} \\ \begin{vmatrix} a' & a'' \\ c' & c'' \end{vmatrix} & \begin{vmatrix} b' & b'' \\ d' & d'' \end{vmatrix} \end{vmatrix}^2 . \tag{3.23}$$

Proof. This can be established by direct computation. Assuming w.l.o.g. that the entries a'', b'',$c''.d''$ are all nonzero, we can take out the common product $(a''b''c''d'')^2$ from all terms on the left-hand side. Denoting all the ratios a/a'', a'/a'' by capitals A, A' etc., and noting that the 2×2 determinant $\begin{vmatrix} a/a'' & 1 \\ b/b'' & 1 \end{vmatrix}$ is simply given by $A - B$ (and in a similar way the other determinants occurring in the expression on the left-hand side), then the left-hand side of (3.23) is representable by

$$a''^2\, b''^2\, c''^2\, d''^2 \Bigg[\left(\begin{vmatrix} A-B & A'-D' \\ C-B & C'-D' \end{vmatrix} + \begin{vmatrix} A'-B' & A-D \\ C'-B' & C-D \end{vmatrix} \right)^2 \\ -4 \begin{vmatrix} A-B & A-D \\ C-B & C-D \end{vmatrix} \cdot \begin{vmatrix} A'-B' & A'-D' \\ C'-B' & C'-D' \end{vmatrix} \Bigg] .$$

Computing the expression between brackets, we observe that it can be simplified to:

$$\big((A-C)(B'-D')+(D-B)(C'-A')\big)^2-4(A-C)(B-D)(A'-C')(B'-D')$$
$$=\left|\begin{array}{cc} A-C & B-D \\ A'-C' & B'-D' \end{array}\right|^2 ,$$

which leads to the desired result. □

This compound theorem to the best of our knowledge is a new result in the theory of hyper-determinants. It seems intimately linked to the structure of the linear equations (the Lax relations) from which it originate in the present context, and there may be analogues for the case of higher rank hyper-determinants (this is currently under investigation). A connection between hyper-determinants and minors of symmetric matrices was established in [14], but it is not clear whether (and if so how) those results are related to the above proposition.

Identifying the coefficients of the system of homogeneous equations (3.19) as entries of a $2\times2\times2$ hyper-determinant, we observe that the structure of this hyper-determinant is exactly of the form as given in Lemma 3.1, and hence we have the following immediate corollary.

Proposition 3.3. *Identifying the 8 entries* $(a_{ijk})_{i,j,k=0,1}$ *by comparing the first two equations of* (3.22) *with the system of equations* (3.19)*, the hyper-determinant takes the form as given by the compound theorem Lemma 3.1, and hence reduces to a perfect square of the form:*

$$\left|\begin{array}{cc} \left|\begin{array}{cc} A_{ilj} & A_{il'j} \\ A_{i''lj} & A_{i''l'j} \end{array}\right| & \left|\begin{array}{cc} A_{i'lj} & A_{i'l'j} \\ A_{i''lj} & A_{i''l'j} \end{array}\right| \\ & \\ \left|\begin{array}{cc} B_{ilj} & B_{il'j} \\ B_{i''lj} & B_{i''l'j} \end{array}\right| & \left|\begin{array}{cc} B_{i'lj} & B_{i'l'j} \\ B_{i''lj} & B_{i''l'j} \end{array}\right| \end{array}\right|^2 \qquad (j=1,2,3), \tag{3.24}$$

where

$$
\begin{vmatrix} A_{ilj} & A_{il'j} \\ A_{i''lj} & A_{i''l'j} \end{vmatrix} = \frac{\sigma(\widehat{\xi}_l - \widehat{\xi}_3)\,\sigma(\widehat{\xi}_{l'} - \widehat{\xi}_3)\,\sigma(\widehat{\xi}_l - \widehat{\xi}_{l'})}{\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_l - \alpha)\,\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_{l'} - \alpha)\,\sigma(\widehat{\widetilde{\xi}}_{i''} - \widehat{\xi}_l - \alpha)\sigma(\widehat{\widetilde{\xi}}_{i''} - \widehat{\xi}_{l'} - \alpha)}
$$
$$
\times \frac{\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\widetilde{\xi}}_{i''})\,\sigma(\widehat{\widetilde{\xi}}_i + \widehat{\widetilde{\xi}}_{i''} - \widehat{\xi}_l - \widehat{\xi}_{l'} - \widehat{\xi}_3 + \xi_j - 2\alpha + \beta)}{\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_3 - \alpha)\,\sigma(\widehat{\widetilde{\xi}}_{i''} - \widehat{\xi}_3 - \alpha)\,\sigma(\widehat{\xi}_l - \xi_j - \beta)\,\sigma(\widehat{\xi}_{l'} - \xi_j - \beta)\,\sigma(\widehat{\xi}_3 - \xi_j - \beta)}, \tag{3.25}
$$

in which we can set $i, i' = 1, 2,\ l, l' = 1, 2 \neq 3$*, and where we naturally should take* $i'' = 3$*. A similar expression for the corresponding determinant in terms of the* B_{ilj} *as given* (3.25) *interchanging* α *and* β *and the shifts* $\widetilde{\ }$ *and* $\widehat{\ }$ *.*

The form (3.25) of the relevant 2×2 determinants, using the expressions for the entries (3.20), is computed in Appendix C.

We apply now the compound theorem Lemma 3.1 to the system of homogeneous equations (3.19). In fact from that system of equations it follows that the ratios $\widehat{h}_i/\widehat{h}_j$ and $\widetilde{k}_i/\widetilde{k}_j$ obey quadratic equations whose discriminant, by virtue of the compound theorem, is a perfect square. Thus, those ratios can be obtained in a rather simple form. We distinguish between the two cases: *i)* the hyper-determinant in question, i.e. the determinant (3.24), vanishes, and *ii)* the hyper-determinant is non-zero.

***i)* Case** (3.24)$= 0$

In this case the resulting set of equations is given by the vanishing of the hyper-determinant, i.e. the set of equations:

$$
\begin{vmatrix} A_{ilj} & A_{il'j} \\ A_{i''lj} & A_{i''l'j} \end{vmatrix} \begin{vmatrix} B_{i'lj} & B_{i'l'j} \\ B_{i''lj} & B_{i''l'j} \end{vmatrix} = \begin{vmatrix} A_{i'lj} & A_{i'l'j} \\ A_{i''lj} & A_{i''l'j} \end{vmatrix} \begin{vmatrix} B_{ilj} & B_{il'j} \\ B_{i''lj} & B_{i''l'j} \end{vmatrix} \tag{3.26}
$$

Inserting the explicit expression (3.25), and its counterpart in terms of the quantities B_{ilj}, into (3.26) we obtain the relations

$$\frac{\sigma(\widehat{\widetilde{\xi}}_i+\widehat{\widetilde{\xi}}_{i''}-\widehat{\xi}_l-\widehat{\xi}_{l'}-\widehat{\xi}_3+\xi_j+\beta-2\alpha)}{\sigma(\widehat{\widetilde{\xi}}_{i'}+\widehat{\widetilde{\xi}}_{i''}-\widehat{\xi}_l-\widehat{\xi}_{l'}-\widehat{\xi}_3+\xi_j+\beta-2\alpha)}\,\frac{\sigma(\widehat{\widetilde{\xi}}_{i'}-\widehat{\xi}_l-\alpha)\sigma(\widehat{\widetilde{\xi}}_{i'}-\widehat{\xi}_{l'}-\alpha)\sigma(\widehat{\widetilde{\xi}}_{i'}-\widehat{\xi}_3-\alpha)}{\sigma(\widehat{\widetilde{\xi}}_i-\widehat{\xi}_l-\alpha)\sigma(\widehat{\widetilde{\xi}}_i-\widehat{\xi}_{l'}-\alpha)\sigma(\widehat{\widetilde{\xi}}_i-\widehat{\xi}_3-\alpha)}$$

$$=\frac{\sigma(\widehat{\widetilde{\xi}}_i+\widehat{\widetilde{\xi}}_{i''}-\widetilde{\xi}_l-\widetilde{\xi}_{l'}-\widetilde{\xi}_3+\xi_j+\alpha-2\beta)}{\sigma(\widehat{\widetilde{\xi}}_{i'}+\widehat{\widetilde{\xi}}_{i''}-\widetilde{\xi}_l-\widetilde{\xi}_{l'}-\widetilde{\xi}_3+\xi_j+\alpha-2\beta)}\,\frac{\sigma(\widehat{\widetilde{\xi}}_{i'}-\widetilde{\xi}_l-\beta)\sigma(\widehat{\widetilde{\xi}}_{i'}-\widetilde{\xi}_{l'}-\beta)\sigma(\widehat{\widetilde{\xi}}_{i'}-\widetilde{\xi}_3-\beta)}{\sigma(\widehat{\widetilde{\xi}}_i-\widetilde{\xi}_l-\beta)\sigma(\widehat{\widetilde{\xi}}_i-\widetilde{\xi}_{l'}-\beta)\sigma(\widehat{\widetilde{\xi}}_i-\widetilde{\xi}_3-\beta)}$$

$$(j=1,2,3)\ , \qquad (3.27)$$

where again we can set $i,i'=1,2$, $l,l'=1,2\neq 3$, and where we naturally should take $i''=3$. The set of relations (3.27) is a coupled system of three quadrilateral equations (for $j=1,2,3$) of 3-leg type, i.e. in terms of three independent variables which reside in the arguments of the Weierstrass σ-functions[3]. We note that all three equations (for $j=1,2,3$) have a common factor, which in the case of a further reduction $\xi_1+\xi_2+\xi_3=0$(mod period lattice) involves only the "long legs" (i.e. the differences over the diagonal). Thus, this system of equations may be too simple to figure as a proper candidate for a higher-rank analogue of the Adler lattice equation.

ii) **Case** (3.24)$\neq 0$

As a consequence of the compound theorem, Lemma 3.1, the hyper-determinant in the case at hand is a perfect square. Thus, going back to the system (3.19), by first eliminating the ratio $\widehat{h}_i/\widehat{h}_j$, we obtain a quadratic for the ratio $\widetilde{k}_i/\widetilde{k}_j$, $(i,j=1,2)$ from which the latter can be solved using the fact that the discriminant of the quadratic (which coincides with the hyper-determinant) is a perfect square. Thus, we get rather manageable expressions for the solutions of the mentioned ratios in terms of the 2×2 determinants involving the expressions A_{ilj} and B_{ilj}. The result of this computation is the following:

Proposition 3.4. *If the expression* (3.24) *is non-vanishing, we have the following solutions of the system* (3.19) *given in terms of the ratios (i.e., up to a common multiplicative*

[3] An equivalent system of equations would have been obtained if, rather than eliminating h_3 and k_3 in its derivation, we would have eliminated one of the other variables among the coefficients h_l and k_l.

factor)

$$\text{either} \qquad \frac{\widehat{h}_1}{\widehat{h}_2} = -\frac{A_{32j}}{A_{31j}} \quad \text{together with} \quad \frac{\widetilde{k}_1}{\widetilde{k}_2} = -\frac{B_{32j}}{B_{31j}}\ , \tag{3.28a}$$

$$\text{or} \qquad \frac{\widehat{h}_1}{\widehat{h}_2} = -\frac{\begin{vmatrix} B_{11j} & A_{12j} & B_{12j} \\ B_{21j} & A_{22j} & B_{22j} \\ B_{31j} & A_{32j} & B_{32j} \end{vmatrix}}{\begin{vmatrix} B_{11j} & A_{11j} & B_{12j} \\ B_{21j} & A_{21j} & B_{22j} \\ B_{31j} & A_{31j} & B_{32j} \end{vmatrix}} \quad \text{together with} \quad \frac{\widetilde{k}_1}{\widetilde{k}_2} = -\frac{\begin{vmatrix} A_{11j} & A_{12j} & B_{12j} \\ A_{21j} & A_{22j} & B_{22j} \\ A_{31j} & A_{32j} & B_{32j} \end{vmatrix}}{\begin{vmatrix} A_{11j} & A_{12j} & B_{11j} \\ A_{21j} & A_{22j} & B_{21j} \\ A_{31j} & A_{32j} & B_{31j} \end{vmatrix}}\ . \qquad (j=1,2,3) \tag{3.28b}$$

The proof, once again, is by direct computation and involves some determinantal manipulations.

The system of equations resulting from (3.28a), inserting the explicit expressions for the quantities A and B from (3.20) reads as follows

$$\frac{\widehat{h}_1}{\widehat{h}_2} = -\frac{\sigma(\widehat{\widehat{\xi}}_3 - \widehat{\xi}_2 - \widehat{\xi}_3 + \xi_j - \alpha + \beta)\,\sigma(\widehat{\widehat{\xi}}_3 - \widehat{\xi}_1 - \alpha)\,\sigma(\widehat{\xi}_1 - \xi_j - \beta)\,\sigma(\widehat{\xi}_2 - \widehat{\xi}_3)}{\sigma(\widehat{\widehat{\xi}}_3 - \widehat{\xi}_1 - \widehat{\xi}_3 + \xi_j - \alpha + \beta)\,\sigma(\widehat{\widehat{\xi}}_3 - \widehat{\xi}_2 - \alpha)\,\sigma(\widehat{\xi}_2 - \xi_j - \beta)\,\sigma(\widehat{\xi}_1 - \widehat{\xi}_3)}, \tag{3.29a}$$

and

$$\frac{\widetilde{k}_1}{\widetilde{k}_2} = -\frac{\sigma(\widehat{\widetilde{\xi}}_3 - \widetilde{\xi}_2 - \widetilde{\xi}_3 + \xi_j + \alpha - \beta)\,\sigma(\widehat{\widetilde{\xi}}_3 - \widetilde{\xi}_1 - \beta)\,\sigma(\widetilde{\xi}_1 - \xi_j - \alpha)\,\sigma(\widetilde{\xi}_2 - \widetilde{\xi}_3)}{\sigma(\widehat{\widetilde{\xi}}_3 - \widetilde{\xi}_1 - \widetilde{\xi}_3 + \xi_j + \alpha - \beta)\,\sigma(\widehat{\widetilde{\xi}}_3 - \widetilde{\xi}_2 - \beta)\,\sigma(\widetilde{\xi}_2 - \xi_j - \alpha)\,\sigma(\widetilde{\xi}_1 - \widetilde{\xi}_3)}. \qquad (j=1,2,3) \tag{3.30a}$$

Inserting the expressions of (3.20) into the system of equations (comprising the equations for different values of $j = 1, 2, 3$) (3.28b) yields a more complicated system of quadrilateral elliptic 3-leg type of equations, which we have so far not been able to simplify[4]. The problem of finding a rational form for the system of equations, as well as verifying their reducibility under the additional constraint $\xi_1+\xi_2+\xi_3 = 0$(mod period lattice) is currently under investigation. We believe that the latter system of equations may correspond to the proper higher-rank analogue of Adler's lattice equation, but further work is needed to underpin that assertion.

[4] Note that the 3×3 determinants in (3.28b) are almost, but not quite, of Frobenius (i.e., elliptic Cauchy) type.

Remark

In this paper we have proposed higher-rank lattice systems which by the construction we think of as natural analogues of Adler's lattice equation in 3-leg form. It is desirable to find their rational forms, similar to those of the Adler equation, i.e., either in the Weierstrass case given by (1.2) or in the Jacobi case (1.8), because in those forms the D_4 symmetries of the equation are manifest. We mention here that the Jacobi form of Adler's equation (1.8) can be written in a remarkably succinct way using spin vectors.

Introducing a "spin matrix" in the following way:

$$\boldsymbol{G}\,\boldsymbol{\sigma}_3\,\boldsymbol{G}^{-1} = \boldsymbol{S}\cdot\boldsymbol{\sigma}\ , \quad \text{where} \quad \boldsymbol{G} = \begin{pmatrix} 1 & 1 \\ u & v \end{pmatrix}\ , \tag{3.31}$$

using the basis of the standard Pauli matrices $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$, we can identify a (normalised) spin vector as

$$\boldsymbol{S}(u,v) = \frac{1}{v-u}\,(uv-1, -i(uv+1), u+v)\ , \quad |\boldsymbol{S}|^2 = \boldsymbol{S}\cdot\boldsymbol{S} = 1\ . \tag{3.32}$$

Such a spin representation has been used in connection with the Landau-Lifschitz equations, cf. e.g. [2]. We have now the following statement:

Proposition 3.5. *Adler's lattice equation in Jacobi form, i.e.* (1.8)*, can be represented in the following spin form:*

$$J_0 + \boldsymbol{S}(v,\widetilde{v})\cdot\boldsymbol{J}\boldsymbol{S}(\widehat{v},\widehat{\widetilde{v}}) = 0\ , \tag{3.33}$$

in which the coefficient (anisotropy parameters) comprising J_0 *and the* 3×3 *diagonal matrix* $\boldsymbol{J} = \mathrm{diag}(J_1, J_2, J_3)$ *are given by*

$$J_0 = \frac{q-r}{2}\ , \quad J_1 = p\frac{1-qr}{2}\ , \quad J_2 = p\frac{1+qr}{2}\ , \quad J_3 = \frac{q+r}{2}\ , \tag{3.34}$$

with $r = (pQ - qP)/(1-p^2q^2)$.

The proof is by direct computation, writing out the components and identifying the various combinations of terms with the ones occurring in (1.8). Obviously, the particular way (3.33) of writing the equation is not unique: it is subject to the D_4 symmetries of the quadrilateral both in how the spin variables depend on the variables v on the vertices and in how the anisotropy parameters depend on the lattice parameters.

This observation suggests that the search for a rational form of higher-rank Adler lattice systems may involve higher spin variables, which are constructed in the following way. Using a basis of GL_3 given by the set of matrices[5] $\{\boldsymbol{I}_{n_1,n_2} \,|\, n_1, n_1 \in \mathbb{Z}_3\}$, where $\boldsymbol{I_n} = \boldsymbol{I}_{n_1,n_2} := \boldsymbol{\Sigma}^{n_1}\boldsymbol{\Omega}^{n_2} = \omega^{n_1 n_2}\boldsymbol{\Omega}^{n_2}\boldsymbol{\Sigma}^{n_1}$ are defined in terms of the elementary matrices

$$\boldsymbol{\Omega} = \begin{pmatrix} 1 & & \\ & \omega & \\ & & \omega^2 \end{pmatrix} , \quad \boldsymbol{\Sigma} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

and where ω is the $3^{\rm d}$ root of unity, $\omega = \exp(2\pi i/3)$. These matrices obey the following relation

$$\boldsymbol{I_n}\,\boldsymbol{I_m} = \omega^{-n_2 m_1}\boldsymbol{I_{n+m}} \ , \quad \text{with} \quad \boldsymbol{n},\boldsymbol{m} \in \mathbb{Z}_3^2 \ , \quad \text{and} \quad \boldsymbol{I}_{\boldsymbol{n}}^{\dagger} = \omega^{-n_1 n_2}\boldsymbol{I}_{-\boldsymbol{n}} \ .$$

where the † means Hermitian conjugation. Thus, introducing the (traceless) spin matrix

$$\boldsymbol{S} := \sum_{\substack{\boldsymbol{n}\in\mathbb{Z}_3^2 \\ \boldsymbol{n}\neq(0,0)}} S_{\boldsymbol{n}}\cdot\boldsymbol{I_n} = \boldsymbol{G}\,\boldsymbol{\Omega}\,\boldsymbol{G}^{-1} \ , \quad \text{with} \quad \boldsymbol{G} = \begin{pmatrix} 1 & 1 & 1 \\ u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \end{pmatrix} , \quad \boldsymbol{\Omega} = \begin{pmatrix} 1 & & \\ & \omega & \\ & & \omega^2 \end{pmatrix} ,$$

which is normalised through identity $\boldsymbol{S}^3 = \boldsymbol{1}$, we can identify an 8-component spin vector $\boldsymbol{S} = (S_{n_1,n_2})$:

in terms of the a 8-component vector $\boldsymbol{S} = (S_{n_1,n_2})$ where $n_1, n_2 = 0, 1, 2$, $(n_1, n_2) \neq (0, 0)$, whose entries can be identified in the following way:

$$\boldsymbol{S} = \frac{1}{\boldsymbol{u}\cdot(\boldsymbol{v}\times\boldsymbol{w})}\ (\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})\,\boldsymbol{\Omega} \begin{pmatrix} (\boldsymbol{v}\times\boldsymbol{w})^T \\ (\boldsymbol{w}\times\boldsymbol{u})^T \\ (\boldsymbol{u}\times\boldsymbol{v})^T \end{pmatrix} ,$$

from which the 8 spin components $S_{n_1,n_2} = S_{n_1,n_2}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})$, $(n_1, n_2 \in \mathbb{Z}_3$, $(n_1, n_2) \neq (0, 0))$, can be inferred comparing the entries of the matrices on the left-hand and right-hand sides. We aim at exploring the possibility of writing the higher-rank lattice systems in this representation.

[5] Following [7] this can obviously be readily generalized to the case of GL_N.

4 Discussion

In this paper we have proposed and investigated a general class of higher-rank elliptic Lax representations for systems of partial difference equations on the 2D lattice. Distinguishing between what we called spin-zero (generalizations of Adler's lattice equation) and spin-nonzero (generalized landau-Lifschitz type) models, we gave the general structure of the resulting equations (from the compatibility conditions) for the latter, but concentrated mainly on the former case for $N = 2$ and $N = 3$. For $N = 2$ we have shown that the Lax systems leads indeed to Adler's lattice equation in its 3-leg form (for the Weierstrass class) and we have analysed how these results generalize to the case $N = 3$ (as a representative example for the higher-rank case). Having established the resulting systems of equations, generalizing Adler's 3-leg form, further work is needed to properly identify those systems. Thus, in further study we will investigate their rational and hyperbolic degenerations, as well as their continuum limits. A possible outcome would be to establish a connection with a differential system obtained by O. Mokhov in the 1980s, [20], arising from third order commuting differential operators defining rank 3 vector bundles over an elliptic curve, cf. [19].

In our view, the significance of the results of this paper is not only to add a new class of elliptic type of integrable systems to our already substantial zoo of such systems, but to depart from the rather restrictive confinement of 2×2 systems to which all ABS type systems, [3], belong. To obtain good insights in the essential structures behind (discrete and continuous) integrable systems, such departures into the multi-component cases are necessary. In the present paper we concentrated mostly on the spin-zero case, while the elaboration of the spin non-zero case is the subject of a future publication, some initial results of which were already presented in [22]. As a direction for the future, establishing connections, if any, with the recently found master-solutions of the quantum Yang-Baxter equations, [6], may be of interest.

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A Proof of the Q4 3-leg identity

The proof of the elliptic identity (3.5) can be achieved directly by showing that the coefficients of each monomials $1, X, Y, Z, XY, XZ, YZ$ and XYZ of the identity are equivalent. By expanding the left-hand side of the identity as

$$\begin{aligned} LHS := {} & (1-t^2)XYZ + (t^2\wp(\xi-\alpha) - \wp(\xi+\alpha))YZ + (t^2\wp(\xi+\beta) - \wp(\xi-\beta))XZ \\ + {} & (t^2\wp(\xi+\alpha-\beta) - \wp(\xi-\alpha+\beta))XY + (\wp(\xi-\beta)\wp(\xi-\alpha+\beta) \\ - {} & t^2\wp(\xi+\beta)\wp(\xi+\alpha-\beta))X + (\wp(\xi+\alpha)\wp(\xi-\alpha+\beta) - t^2\wp(\xi-\alpha)\wp(\xi+\alpha-\beta))Y \\ + {} & (\wp(\xi+\alpha)\wp(\xi-\beta) - t^2\wp(\xi-\alpha)\wp(\xi+\beta))Z + t^2\wp(\xi-\alpha)\wp(\xi+\alpha-\beta)\wp(\xi+\beta) \\ - {} & \wp(\xi+\alpha)\wp(\xi-\beta)\wp(\xi-\alpha+\beta)\ , \end{aligned} \tag{A.35}$$

it is obvious that the first term of line 1 is equal to the corresponding term on the right hand-side of (3.5). The rest of the equalities of the corresponding coefficients follow by the same method as explained below. First, we make use of the Frobenius-Stickelberger formula [10] stated in Appendix B, in terms of the variables $(\xi, \alpha, -\beta)$

$$\begin{vmatrix} 1 & \wp(\xi) & \wp'(\xi) \\ 1 & \wp(\alpha) & \wp'(\alpha) \\ 1 & \wp(-\beta) & \wp'(-\beta) \end{vmatrix} = 2\frac{\sigma(\xi+\alpha-\beta)\,\sigma(\xi-\alpha)\,\sigma(\alpha+\beta)\,\sigma(\xi+\beta)}{\sigma^3(\xi)\,\sigma^3(\alpha)\,\sigma^3(\beta)},$$

and a similar relation with $(\xi, -\alpha, \beta)$. If we divide the former determinant by the latter one, we obtained the following expression for t and s in (3.6)

$$t = \frac{\wp'(\xi)(b-a) - Ab - aB + \wp(\xi)(A+B)}{\wp'(\xi)(b-a) + Ab + aB - \wp(\xi)(A+B)},\ s = \frac{4(a-b)\wp'(\xi)}{(\wp'(\xi)(b-a) + Ab + aB - \wp(\xi)(A+B))^2}\ .$$

Applying the elliptic addition formulae of the form, notably:

$$\wp(\xi)+\wp(\eta)+\wp(\xi\pm\eta)=\frac{1}{4}\left(\frac{\wp'(\xi)\mp\wp'(\eta)}{\wp(\xi)-\wp(\eta)}\right)^2 , \tag{A.36}$$

on (A.35), we get on the one hand

$$\begin{aligned}
LHS &= (1-t^2)XYZ+(a+\wp(\xi)-\frac{(\wp'(\xi)-A)^2}{4(\wp(\xi)-a)^2}+t^2(-a-\wp(\xi)+\frac{(\wp'(\xi)+A)^2}{4(\wp(\xi)-a)^2}))YZ\\
&+ (b+\wp(\xi)-\frac{(\wp'(\xi)+B)^2}{4(\wp(\xi)-b)^2}+t^2(-b-\wp(\xi)+\frac{(\wp'(\xi-B)^2}{4(\wp(\xi)-b)^2}))XZ\\
&+ (c+\wp(\xi)-\frac{(\wp'(\xi)-C)^2}{4(\wp(\xi)-c)^2}+t^2(-c-\wp(\xi)+\frac{(\wp'(\xi)+C)^2}{4(\wp(\xi)-c)^2}))XY\\
&+ ((-a-\wp(\xi)+\frac{(\wp'(\xi)-A)^2}{4(\wp(\xi)-a)^2})(-b-\wp(\xi)+\frac{(\wp'(\xi)+B)^2}{4(\wp(\xi)-b)^2})\\
&- t^2((-a-\wp(\xi)+\frac{(\wp'(\xi)+A)^2}{4(\wp(\xi)-a)^2})(-b-\wp(\xi)+\frac{(\wp'(\xi)-B)^2}{4(\wp(\xi)-b)^2})))Z\\
&+ ((-a-\wp(\xi)+\frac{(\wp'(\xi)-A)^2}{4(\wp(\xi)-a)^2})(-c-\wp(\xi)+\frac{(\wp'(\xi)-C)^2}{4(\wp(\xi)-c)^2})\\
&- t^2((-a-\wp(\xi)+\frac{(\wp'(\xi)+A)^2}{4(\wp(\xi)-a)^2})(-c-\wp(\xi)+\frac{(\wp'(\xi)+C)^2}{4(\wp(\xi)-c)^2})))Y\\
&+ ((-b-\wp(\xi)+\frac{(\wp'(\xi)+B)^2}{4(\wp(\xi)-b)^2})(-c-\wp(\xi)+\frac{(\wp'(\xi)-C)^2}{4(\wp(\xi)-c)^2})\\
&- t^2((-b-\wp(\xi)+\frac{(\wp'(\xi)-B)^2}{4(\wp(\xi)-b)^2})(-c-\wp(\xi)+\frac{(\wp'(\xi)+C)^2}{4(\wp(\xi)-c)^2})))X\\
&+ ((a+\wp(\xi)-\frac{(\wp'(\xi)-A)^2}{4(\wp(\xi)-a)^2})(-b-\wp(\xi)+\frac{(\wp'(\xi)+B)^2}{4(\wp(\xi)-b)^2})(-c-\wp(\xi)\\
&+ \frac{(\wp'(\xi)-C)^2}{4(\wp(\xi)-c)^2})+t^2((-a-\wp(\xi)+\frac{(\wp'(\xi)+A)^2}{4(\wp(\xi)-a)^2})(-b-\wp(\xi)\\
&+ \frac{(\wp'(\xi)-B)^2}{4(\wp(\xi)-b)^2})(-c-\wp(\xi)+\frac{(\wp'(\xi)+C)^2}{4(\wp(\xi)-c)^2}))).
\end{aligned} \tag{A.37}$$

The proof is completed by using the relations (1.5) and subsequently (1.3), (1.4) on the terms of (A.37) and as well as on the right hand-side of (3.5).

B Frobenius-Stickelberger type identities

Here we collect a number of results related to elliptic determinantal formulae of Frobenius and Frobenius-Stickelberger type (i.e. elliptic Cauchy and Vandermonde determinants).

The Frobenius-Stickelberger formula, [10] is given by

$$\begin{vmatrix} 1 & \wp(x_1) & \wp'(x_1) & \cdots & \wp^{(n-2)}(x_1) \\ 1 & \wp(x_2) & \wp'(x_2) & \cdots & \wp^{(n-2)}(x_2) \\ 1 & \wp(x_3) & \wp'(x_3) & \cdots & \wp^{(n-2)}(x_3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \wp(x_n) & \wp'(x_n) & \cdots & \wp^{(n-2)}(x_n) \end{vmatrix}$$
$$= \quad (-1)^{(n-1)(n-2)/2}\ 1!2!3!...(n-1)!\ \frac{\sigma(x_1+x_2+...+x_n)\prod_{i<j=1}^{n}\sigma(x_i-x_j)}{\prod_{i=1}^{n}\sigma^n(x_i)} \tag{B.1}$$

Denoting the Frobenius-Stickelberger *matrix* $\mathcal{P}(x_0,x_1,\ldots,x_n)=\mathcal{P}(\boldsymbol{x})$ by:

$$\mathcal{P}(\boldsymbol{x}) = \begin{pmatrix} 1 & \wp(x_1) & \wp'(x_1) & \cdots & \wp^{(n-2)}(x_1) \\ 1 & \wp(x_2) & \wp'(x_2) & \cdots & \wp^{(n-2)}(x_2) \\ 1 & \wp(x_3) & \wp'(x_3) & \cdots & \wp^{(n-2)}(x_3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \wp(x_n) & \wp'(x_n) & \cdots & \wp^{(n-2)}(x_n) \end{pmatrix} \tag{B.2}$$

we have by using Cramer's rule the following factorisation formula:

$$\left[\mathcal{P}(\boldsymbol{x})\cdot\mathcal{P}(\boldsymbol{y})^{-1}\right]_{i,j} = \frac{1}{\sigma^n(x_i)}\,\Phi_\Sigma(x_i-y_j)\sigma^n(y_j)\frac{\prod_{l=1}^{n}\sigma(x_i-y_l)}{\prod_{l\neq j}\sigma(y_j-y_l)}\ , \tag{B.3}$$

in which $\Sigma\equiv\Sigma_{l=1}^{n}y_l$. As a consequence we obtain from this the Frobenius determinantal formula, [11]

$$\det\left(\Phi_\kappa(x_i-y_j)\right)_{i,j=1,\ldots,N} = \frac{\sigma(\kappa+\Sigma)}{\sigma(\kappa)}\,\frac{\prod_{i<j}\ \sigma(x_i-x_j)\,\sigma(y_j-y_i)}{\prod_{i,j}\ \sigma(x_i-y_j)}\ ,\quad \Sigma:=\sum_{i=1}^{N}(x_i-y_i)\ . \tag{B.4}$$

Conversely, the Frobenius-Stickelberger formula (B.1) can be obtained from the Frobenius formula by a set of degenerate limits. The elliptic Lagrange interpolation formulae

$$\prod_{i=1}^{N}\frac{\sigma(\xi-x_i)}{\sigma(\xi-y_i)} = \sum_{i=1}^{N}\Phi_{-\Sigma}(\xi-y_i)\frac{\prod_{j=1}^{N}\sigma(y_i-x_j)}{\prod_{\substack{j=1\\ j\neq i}}^{N}\sigma(y_i-y_j)}\ , \tag{B.5}$$

which holds if $\Sigma\neq 0$, and if $\Sigma=0$:

$$\prod_{i=1}^{N}\frac{\sigma(\xi-x_i)}{\sigma(\xi-y_i)} = \sum_{i=1}^{N}\left[\zeta(\xi-y_i)-\zeta(x-y_i)\right]\frac{\prod_{j=1}^{N}\sigma(y_i-x_j)}{\prod_{\substack{j=1\\ j\neq i}}^{N}\sigma(y_i-y_j)}\ , \tag{B.6}$$

where x denotes any of the zeroes x_i, $(i = 1, \dots, N)$. Both (B.5) can be obtained from the Frobenius formula [11] by row-or column expansions (adding an extra row and column to the Frobenius matrix, say with $x_0 = \xi$ and $y_0 = \eta$, and then expanding along that row or column) and (B.6) can subsequently be obtained from a limiting case of the latter.

C Proof of Equation (3.25)

Here, we present the proof of the determinant in (3.25). By definition of A_{ilj} given in (3.20) we have

$$\begin{vmatrix} A_{ilj} & A_{il'j} \\ A_{i'lj} & A_{i'l'j} \end{vmatrix} = \frac{\sigma(\widehat{\xi}_l - \widehat{\xi}_3)\sigma(\widehat{\xi}_{l'} - \widehat{\xi}_3)}{S(\widehat{\widehat{\xi}}_i)\, S(\widehat{\widehat{\xi}}_{i'})\sigma(\widehat{\xi}_l - \xi_j - \beta)\sigma(\widehat{\xi}_{l'} - \xi_j - \beta)\sigma^2(\widehat{\xi}_3 - \xi_j - \beta)}$$
$$\Big[\sigma(\widehat{\widehat{\xi}}_i - \widehat{\xi}_3 - \widehat{\xi}_l + \xi_j - \alpha + \beta)\sigma(\widehat{\widehat{\xi}}_{i'} - \widehat{\xi}_3 - \widehat{\xi}_{l'} + \xi_j - \alpha + \beta)\sigma(\widehat{\widehat{\xi}}_i - \widehat{\xi}_{l'} - \alpha)\sigma(\widehat{\widehat{\xi}}_{i'} - \widehat{\xi}_l - \alpha)$$
$$-\sigma(\widehat{\widehat{\xi}}_{i'} - \widehat{\xi}_3 - \widehat{\xi}_l + \xi_j - \alpha + \beta)\sigma(\widehat{\widehat{\xi}}_i - \widehat{\xi}_3 - \widehat{\xi}_{l'} + \xi_j - \alpha + \beta)\sigma(\widehat{\widehat{\xi}}_i - \widehat{\xi}_l - \alpha)\sigma(\widehat{\widehat{\xi}}_{i'} - \widehat{\xi}_{l'} - \alpha)\Big], \tag{C.1}$$

where

$$S(\xi) = \sigma(\xi - \widehat{\xi}_l - \alpha)\sigma(\xi - \widehat{\xi}_{l'} - \alpha)\sigma(\xi - \widehat{\xi}_3 - \alpha).$$

Noting that the difference in the bracket can be simplified by applying the three-term relation for the σ-function in the following form:

$$\begin{aligned} \sigma(x-a)\sigma(y-b)\sigma(z-b)\sigma(w-a) \quad &- \quad \sigma(y-a)\sigma(x-b)\sigma(z-a)\sigma(w-b) \\ &= \quad \sigma(z+y-a-b)\sigma(x-y)\sigma(x-z)\sigma(b-a), \end{aligned} \tag{C.2}$$

in which $x - y = z - w$. Making now the the following choice for x, y, z, w, a and b in the identity (C.2):

$$\begin{aligned} x &= \widehat{\widehat{\xi}}_i - \widehat{\xi}_3 + \xi_j - \alpha + \beta & y &= \widehat{\widehat{\xi}}_{i'} - \widehat{\xi}_3 + \xi_j - \alpha + \beta \\ z &= \widehat{\widehat{\xi}}_i - \alpha & w &= \widehat{\widehat{\xi}}_{i'} - \alpha \\ a &= \widehat{\xi}_l & b &= \widehat{\xi}_{l'} \end{aligned}$$

the expression between brackets on the right-hand side of (C.1) simplifies to

$$[\cdots] = \sigma(-\widehat{\xi}_3 + \xi_j + \beta)\, \sigma(\widehat{\widehat{\xi}}_i + \widehat{\widehat{\xi}}_{i'} - \widehat{\xi}_l - \widehat{\xi}_{l'} - \widehat{\xi}_3 + \xi_j - 2\alpha + \beta)\, \sigma(\widehat{\widehat{\xi}}_i - \widehat{\widehat{\xi}}_{i'})\, \sigma(\widehat{\xi}_{l'} - \widehat{\xi}_l)\, . \tag{C.3}$$

Substituting the right-hand side of (C.3) into (C.1) and cancelling the first factor against the corresponding factor in the prefactor of (C.1), using the fact that σ is an odd function, we obtaine the desired result given by the determinant in (3.25). In a similar way (or by making the obvious replacements $\alpha \leftrightarrow \beta$ and $\tilde{} \leftrightarrow \hat{}$) the computation of the 2×2 determinant B_{ilj} can be verified.

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